Exam Complex Analysis April 17, 2012

The exam consists of 5 problems. You should **write clearly, argue clearly, and motivate your answers**. Points available can be found below.

1. Consider the complex function $f(z) = x^3 - 3y^2 + 2x + i(3x^2y - y^3 + 2y)$ for $z = x + iy$.

   a. Determine all points in $\mathbb{C}$ where $f(z)$ is differentiable. Clearly motivate your answer!

   b. Show that $f(z)$ is nowhere analytic.

2. Consider the function $g(z) = \dfrac{e^{\pi z}}{4z^2 + 1}$.

   a. Let Γ_1 be the circle $|z| = 2$, traversed once in counterclockwise direction. Determine $\int_{\Gamma_1} g(z)dz$.

   b. Let Γ_2 be the circle $|z| = \frac{1}{4}$, traversed once in counterclockwise direction. Compute $\int_{\Gamma_2} g(z)dz$.

3. Let $f(z)$ be analytic on $|z| \leq 1$. Let C be the circle $|z| = 1$, traversed once in the counterclockwise direction.

   a. Determine $\int_C \left(\dfrac{1}{z} - f(z)\right) dz$.

   b. Prove that $\max_{|z|=1} |\dfrac{1}{z} - f(z)| \geq 1$. (Hint: find an upper bound for the integral appearing in part (a.)).

4. Let $f(z)$ be the function given by $f(z) = \tan z$.

   a. Determine the zeros of $f(z)$ and their orders.

   b. Determine the singularities of $f(z)$.

   c. What kind of singularity does $f(z)$ have at $z = \frac{1}{2}\pi$?

   d. Determine the residue of $f(z)$ at $z = \frac{1}{2}\pi$.

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5. Rouché's theorem is a very powerful result to obtain information about the zeros of analytic functions.

a. Give a precise formulation of Rouché's theorem.

b. Consider the equation $2z^5 + 8z - 1 = 0$. Determine the number of roots of the equation in the disc $|z| < 2$.

c. Show that the above equation has exactly one root in the disc $|z| < 1$, and that this root real and positve.

Points:

Problem 1: 16
Problem 2: 18
Problem 3: 16
Problem 4: 18
Problem 5: 22
10 points for free